MEMORANDUM
RM-2993-PR
APRIL 1962

A LINEAR PROGRAM OF PRAGER'S
Notes on Linear Programming and Extensions – Part 60

Oliver Gross

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PREFACE

An infinite-dimensional linear-programming problem proposed by W. Prager is solved in this Memorandum, which involves basic mathematical research. The problem arose in an elastico-plastic structural-design context. It is hoped that the solution of this problem will provide insight into the solution of many others of the same general character.

SUMMARY

For the problem of minimizing the integral $\int_0^1 f(x)dx$ subject to the constraints

$$- f(x) \leq xg(y) \leq f(x) \qquad \text{if } 0 \leq x \leq y \leq 1,$$
$$- f(x) \leq xg(y) - x + y \leq f(x) \qquad \text{if } 0 \leq y \leq x \leq 1,$$

the author exhibits solutions and proves both that they satisfy the constraints and that they have the extremizing property.

The problem, proposed by W. Prager, arose in an elastico-plastic, structural-design context.

CONTENTS

A LINEAR PROGRAM OF PRAGER'S

1. STATEMENT OF THE PROBLEM AND OF ITS SOLUTION

In a recent letter to R. Bellman, W. Prager proposed the following problem:

It is required to find real-valued, nonnegative integrable functions f, g, on [0, 1] so as to minimize

$$\int_0^1 f(x)dx,$$

subject to the constraints

$$-f(x) \le xg(y) \le f(x) \qquad \text{if } 0 \le x \le y \le 1,$$
$$-f(x) \le xg(y) - x + y \le f(x) \qquad \text{if } 0 \le y \le x \le 1.$$

The foregoing problem arose in a structural-design context of elastico-plastic type. Although the writer is not familiar with the details of the model, it is hoped that the solution of the present problem will provide insight toward the solution of an entire class of problems of the same general character.

The solution (f_0, g_0) given below is not the only solution to the problem, as the subsequent analysis indicates. It turns out that g_0 can be perturbed slightly in its tail (beyond $\sqrt{2}/2$) without altering feasibility. Moreover, since the other component (f_0) is defined only in terms of the values of g_0 on the interval $[0, \sqrt{2}/2]$, integrability of g_0 beyond $\sqrt{2}/2$ is inessential. It is thus seen that there are 2^c solutions of the problem, where c is the cardinality of the continuum.

Finally we remark that, corresponding to an optimal (f, g) pair, if $\bar{f} \geq f$, with equality holding almost everywhere, then $(\bar{f}, g)$ is an optimal feasible solution to the problem.

Having dispensed with this digression, we now contend that the following (f_0, g_0) is a solution to Prager's problem:

$$g_0(x) = \max \left[0, \lambda - \log \left(x + \sqrt{x^2 + \tfrac{1}{2}}\right)\right],$$

where

$$\lambda = \frac{1}{2}\left[1 + \log \left(\frac{\sqrt{2} + 1}{2}\right)\right],$$

and

$$f_0(x) = \begin{cases} xg_0(x) & \text{if } 0 \leq x < \frac{\sqrt{2}}{2}, \\ -xg_0\left(\sqrt{x^2 - \frac{1}{2}}\right) + x - \sqrt{x^2 - \frac{1}{2}} & \text{if } \frac{\sqrt{2}}{2} \leq x \leq 1. \end{cases}$$

In Sec. 2 we shall show that (f_0, g_0) satisfies the constraints imposed. In the final Sec. 3 we shall then show that (f_0, g_0) minimizes the objective integral over all pairs (f, g) satisfying the constraints.

2. VERIFICATION OF FEASIBILITY OF SOLUTION

It is clear that the given constraints can be rewritten as follows:

$$g(x) \geq 0, \tag{A}$$

$$f(x) \geq 0, \tag{B}$$

$$f(x) \geq xg(y) \qquad \text{if } 0 \leq x \leq y, \tag{C}$$

$$f(x) \geq xg(y) - x + y \qquad \text{if } 0 \leq y \leq x, \tag{D}$$

$$f(x) \geq -xg(y) + x - y \qquad \text{if } 0 \leq y \leq x. \tag{E}$$

We remark that the given constraint

$$-f(x) \leq xg(y) \qquad \text{if } 0 \leq x \leq y$$

is a trivial consequence of the nonnegativity conditions, (A) and (B).

Before we show that the given pair (f_0, g_0) satisfies the constraints (A) ... (E), a few preliminary remarks are in order.

Remark 1. g_0 is decreasing and convex.

To see this it is sufficient to show that on the set where $g_0 > 0$, its derivative is negative and increasing. But on this set, we have

$$(1) \qquad g_0'(x) = -\frac{1}{\sqrt{x^2 + \frac{1}{2}}} .$$

It is immediate that the right-hand member of (1) has the requisite properties.

Remark 2. $g_0(0) < 1$.

A numerical calculation gives

$$g_0(0) \approx .9407 < 1.$$

Remark 3. $g_0(x) < 1$.

This is an immediate consequence of the two preceding remarks.

Remark 4. $g_0\left(\frac{\sqrt{2}}{2}\right) > 0$.

A numerical calculation gives

$$g_0\left(\frac{\sqrt{2}}{2}\right) \approx .0593 > 0.$$

We now proceed with the proof of feasibility of (f_0, g_0).

That g_0 satisfies the constraint (A) is obvious from the definition of g_0.

Moreover, it follows from (A) and the definition of f_0 that (B) is satisfied by f_0 on the interval $[0, \sqrt{2}/2)$. To see that f_0 is nonnegative on the remaining interval, it suffices to show that

$$f_0(x) = -xg_0\left(\sqrt{x^2 - \tfrac{1}{2}}\right) + x - \sqrt{x^2 - \tfrac{1}{2}} \geq 0 \quad \text{if } x \geq \frac{\sqrt{2}}{2}.$$

But, at $x = \sqrt{2}/2$, the right-hand member of the above equation has the value

$$-\frac{\sqrt{2}}{2}\, g_0(0) + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}\left(1 - g_0(0)\right) > 0,$$

by virtue of remark 2. Consequently (B) will be established if we can show that f is increasing on the interval $[\sqrt{2}/2, 1]$. But, since the function $\sqrt{x^2 - 1/2}$ maps the interval $(\sqrt{2}/2, 1)$ into the interval $(0, \sqrt{2}/2)$, and g_0 satisfies $g_0 > 0$ on this latter interval by virtue of remarks 1 through 4, we may differentiate $f_0(x)$ using

$$g_0(x) = \lambda - \log\left(x + \sqrt{x^2 + \tfrac{1}{2}}\right).$$

Thus, we have

$$f_0'(x) = -xg_0'\left(\sqrt{x^2 - \tfrac{1}{2}}\right)\frac{d}{dx}\sqrt{x^2 - \tfrac{1}{2}} - g_0\left(\sqrt{x^2 - \tfrac{1}{2}}\right)$$
$$+ 1 - \frac{d}{dx}\sqrt{x^2 - \tfrac{1}{2}}.$$

But, from (1) we see that

$$g_0'\left(\sqrt{x^2 - \frac{1}{2}}\right) = -\frac{1}{x} .$$

Substituting this value in the above formula for f_0' gives

$$f_0'(x) = 1 - g_0\left(\sqrt{x^2 - \frac{1}{2}}\right) > 0 ,$$

by virtue of remark 3. This establishes the desired monotonicity of f_0. Hence the constraint (B) is satisfied by (f_0, g_0).

To see that (C) is satisfied by the proposed solution, we must prove that

$$f_0(x) \geq xg_0(y) \qquad \text{if } 0 \leq x \leq y.$$

But, by remark 1, g_0 is decreasing. Hence the right-hand member of the desired relation will be maximal at $y = x$, and thus it is sufficient to establish that

$$(2) \quad f_0(x) \geq xg_0(x) \qquad \text{for all } x \in [0, 1].$$

But this last relation holds with equality on the interval $[0, \sqrt{2}/2)$, by the definition of f_0 on this interval. Consequently, we need to show only that the relation holds on the interval $[\sqrt{2}/2, 1]$, i.e., from the definition of f_0 there, that

$$f_0(x) = -xg_0\left(\sqrt{x^2 - \frac{1}{2}}\right) + x - \sqrt{x^2 - \frac{1}{2}} \geq xg_0(x) \qquad \text{if } \frac{\sqrt{2}}{2} \leq x \leq 1.$$

On the set on which $g_0(x) = 0$, this last relation holds automatically since by (B) we have $f_0(x) \geq 0$. Consequently, we need verify the above only for points for which $g_0(x) > 0$

and $\sqrt{2}/2 \leq x \leq 1$. But at $x = \sqrt{2}/2$, we have

$$f_0\left(\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2} g_0(0) + \frac{\sqrt{2}}{2}$$
$$= -\frac{\sqrt{2}}{2}\left\{\frac{1}{2}\left[1 + \log\left(\frac{\sqrt{2}+1}{2}\right)\right] - \log\left(\frac{\sqrt{2}}{2}\right)\right\}$$
$$+ \frac{\sqrt{2}}{2} \quad .$$

Upon simplification, using well-known properties of the logarithm and a little algebra, we obtain

$$f_0\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4}\log(\sqrt{2}+1) \quad .$$

On the other hand, the right-hand member of the desired inequality yields, at $x = \sqrt{2}/2$,

$$\frac{\sqrt{2}}{2} g_0\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}\left\{\frac{1}{2}\left[1 + \log\left(\frac{\sqrt{2}+1}{2}\right)\right] - \log\left(\frac{\sqrt{2}}{2}+1\right)\right\}$$
$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4}\log\left(\sqrt{2}+1\right) \quad .$$

Thus, the desired inequality is sharp at $x = \sqrt{2}/2$. To establish it beyond this point on the set on which $g_0 > 0$, it suffices to show that the function

$$f_0(x) - xg_0(x) = -xg_0\left(\sqrt{x^2 - \tfrac{1}{2}}\right) + x - \sqrt{x^2 - \tfrac{1}{2}} - xg_0(x)$$

is increasing on this set.

Upon differentiating, we obtain

$$\left[f_0(x) - xg_0(x)\right]' =$$

$$- xg_0'\left(\sqrt{x^2 - \frac{1}{2}}\right)\frac{d}{dx}\sqrt{x^2 - \frac{1}{2}} + 1 - \frac{d}{dx}\sqrt{x^2 - \frac{1}{2}}$$

$$- xg_0'(x) - g_0(x) \quad .$$

By the previous relation,

$$g_0'\left(\sqrt{x^2 - 1/2}\right) = - 1/x,$$

this reduces to

$$\left(f_0(x) - xg_0(x)\right)' = \left(1 - g_0(x)\right) - xg_0'(x) \quad .$$

But by remark 3, we have $1 - g_0(x) > 0$, and by remark 1, we have $- xg_0'(x) > 0$, whence

$$\left[f_0(x) - xg_0(x)\right]' > 0.$$

Thus (C) is established, and we have moreover also shown that

(2) $\qquad f_0(x) \geq xg_0(x) \qquad\qquad$ for all $x \in [0, 1]$.

To establish (D), we are required to show that

$$f_0(x) \geq xg_0(y) - x + y \qquad \text{if } 0 \leq y \leq x \quad .$$

By remark 1, g_0 is convex. Thus, the right-hand member of the desired inequality is convex in y and hence is maximal at an end point $y = 0$ or $y = x$. Hence it suffices to establish

the inequality at each of these points. But at $y = 0$, the right-hand member becomes $x \left[g_0(0) - 1\right] \leq 0$, by remark 2. But since $f_0(x) \geq 0$ we see that the inequality is satisfied at the lower end point. At $y = x$ the inequality reduces to

$$f_0(x) \geq xg_0(x) \quad .$$

But this last is the inequality (2) already established. Thus condition (D) is satisfied.

Finally, to establish (E) we are required to show that

$$f_0(x) \geq -xg_0(y) + x - y \qquad\qquad \text{if } 0 \leq y \leq x.$$

By remark 1, g_0 is convex; hence the right-hand member of the desired inequality is <u>concave</u> in y. Consequently, if there exists an interior point $y_0 \in (0, x)$ at which the derivative of the right-hand member with respect to y vanishes, this member will be maximal at that point. Upon differentiating with respect to y and equating to zero, we obtain

$$-xg_0'(y) - 1 = 0,$$

or

$$g_0'(y) = -\frac{1}{x} \; .$$

But recall that this equation is satisfied by

$$y_0 = \sqrt{x^2 - \frac{1}{2}} < x,$$

provided of course that $\sqrt{2}/2 \leq x \leq 1$. Moreover, y_0 is in the

range of $g_0 > 0$. Thus for $x \in [\sqrt{2}/2, 1]$ it is enough to show that

$$f_0(x) \geq - xg_0\left(\sqrt{x^2 - \frac{1}{2}}\right) + x - \sqrt{x^2 - \frac{1}{2}} .$$

On this interval, however, equality holds by the definition of f_0. Thus it remains to show that the desired inequality holds on the interval $0 \leq x < \sqrt{2}/2$; i.e., we are required to show that

$$xg_0(x) \geq - xg_0(y) + x - y \qquad \text{if } 0 \leq y \leq x < \frac{\sqrt{2}}{2} .$$

Now since g_0 is strictly convex on the set on which $g_0 > 0$, and $\sqrt{x^2 - 1/2}$ is imaginary for $x < \sqrt{2}/2$, we see the derivative of the right-hand member (with respect to y) cannot vanish at a point y_0 interior to $[0, \sqrt{2}/2)$. Thus the right-hand member is maximal at an end point $y = 0$ or $y = x$. But at $y = 0$, this member reduces to $- xg_0(0) + x$, and we are required to show that

$$xg_0(x) \geq - xg_0(0) + x \qquad \text{if } 0 \leq x < \frac{\sqrt{2}}{2} .$$

Since this inequality holds trivially at $x = 0$, we can assume that $x > 0$, and this last desired inequality becomes

$$g_0(x) \geq - g_0(0) + 1 \qquad \text{if } 0 \leq x < \frac{\sqrt{2}}{2} .$$

But since g_0 is decreasing and continuous on this interval, it suffices to establish the inequality at $x = \sqrt{2}/2$. From a previous calculation, however, we have

$$g_0\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2} - \frac{1}{2}\log(\sqrt{2}+1),$$

$$-g_0(0) + 1 = \frac{1}{2} - \frac{1}{2}\log(\sqrt{2}+1)\ .$$

The foregoing inequality is thus sharp at $y = 0$. Finally, at the upper end point $y = x$, the inequality

$$xg_0(x) \geq -xg_0(y) + x - y$$

reduces to the nonnegativity condition on g_0 already established. Thus, (E) is satisfied and the verification that (f_0, g_0) is in the constraint set is complete.

3. PROOF OF OPTIMALITY OF SOLUTION

The proof of optimality is not nearly so tedious as the rather messy analysis of Sec. 2, which showed that the proposed solution (f_0, g_0) is in the constraint set. The reason for this unbalance of effort is due primarily to the 2-dimensional character of the constraining inequalities and the fact (which we do not prove here) that equality of the constraints can hold only on a set of (2-dimensional) measure zero. This last assertion is an aside, however, and we shall turn our attention now to establishing the minimal property of (f_0, g_0).

Let

$$J = \int_{\frac{\sqrt{2}}{2}}^{1}\left(x - \sqrt{x^2 - \frac{1}{2}}\right)dx.$$

By elementary calculus and a numerical computation, we obtain

$$J = \frac{1}{4}\left(1 + \log(\sqrt{2} + 1) - \sqrt{2}\right) \approx .1168.$$

Moreover, we have

$$f_0(x) = xg_0(x) \qquad \text{if } 0 \le x < \frac{\sqrt{2}}{2},$$

whence

$$(3) \qquad \int_0^{\frac{\sqrt{2}}{2}} f_0(x)dx = \int_0^{\frac{\sqrt{2}}{2}} xg_0(x)dx.$$

Also, we have

$$f_0(x) = -xg_0\left(\sqrt{x^2 - \tfrac{1}{2}}\right) + x - \sqrt{x^2 - \tfrac{1}{2}} \qquad \text{if } \frac{\sqrt{2}}{2} \le x \le 1,$$

whence

$$(4) \qquad \int_{\frac{\sqrt{2}}{2}}^1 f_0(x)dx = -\int_{\frac{\sqrt{2}}{2}}^1 xg_0\left(\sqrt{x^2 - \tfrac{1}{2}}\right)dx + J .$$

Adding equations (3) and (4) together gives

$$\int_0^1 f_0(x)dx = \int_0^{\frac{\sqrt{2}}{2}} xg_0(x)dx - \int_{\frac{\sqrt{2}}{2}}^1 xg_0\left(\sqrt{x^2 - \tfrac{1}{2}}\right)dx + J .$$

Upon making the change of variable

$$x' = \sqrt{x^2 - \tfrac{1}{2}}$$

in the second integral of the right-hand member of the above equation, we obtain

$$\int_{\frac{\sqrt{2}}{2}}^{1} xg_0\left(x^2 - \frac{1}{2}\right) dx = \int_0^{\frac{\sqrt{2}}{2}} x'g_0(x')dx' .$$

Thus, the integrals involving g_0 cancel and we are left with

(5) $$\int_0^1 f_0(x)dx = J .$$

Next, let (f, g) be any pair of functions in the constraint set. Then (C) must be satisfied in particular for $y = x$ and $0 \leq x \leq \sqrt{2}/2$. Thus, we have

(6) $$f(x) \geq xg(x) \qquad \text{if } 0 \leq x \leq \frac{\sqrt{2}}{2},$$

whence we obtain

(7) $$\int_0^{\frac{\sqrt{2}}{2}} f(x)dx \geq \int_0^{\frac{\sqrt{2}}{2}} xg(x)dx .$$

Moreover, if $\sqrt{2}/2 \leq x \leq 1$, we have $0 \leq \sqrt{x^2 - 1/2} < x$, so that (E) must be satisfied if $y = \sqrt{x^2 - 1/2}$ and $\sqrt{2}/2 \leq x \leq 1$; that is,

(8) $$f(x) \geq - xg\left(\sqrt{x^2 - \frac{1}{2}}\right) + x - \sqrt{x^2 - \frac{1}{2}} \quad \text{if } \frac{\sqrt{2}}{2} \leq x \leq 1.$$

Integrating this last inequality over the interval on which it holds yields

(9) $$\int_{\frac{\sqrt{2}}{2}}^{1} f(x)dx \geq -\int_{\frac{\sqrt{2}}{2}}^{1} xg\left(\sqrt{x^2 - \frac{1}{2}}\right) dx + J.$$

Finally, adding (7) and (9) gives

$$\int_0^1 f(x)dx \geq \int_0^{\frac{\sqrt{2}}{2}} xg(x)dx - \int_{\frac{\sqrt{2}}{2}}^{1} xg\left(\sqrt{x^2 - \frac{1}{2}}\right) dx + J.$$

But upon making the same change of variable as before, we see that the integrals involving g cancel and we are left with

$$\int_0^1 f(x)dx \geq J = \int_0^1 f_0(x)dx \qquad \text{(from (5))},$$

whence the minimal property of (f_0, g_0) is established.

A final closing note is in order. In Prager's original statement of the problem it was tacitly assumed that f is integrable (via the objective function). No such restriction (as we have made) was asserted about g. Nevertheless, a proof of optimality with this restriction relaxed can be given as follows:

If we replace x in (8) by $\sqrt{x^2 + 1/2}$, then the new variable ranges over the interval $[0, \sqrt{2}/2]$ and we obtain in place of (8) the equivalent inequality

$$(10) \quad f\left(\sqrt{x^2 + \tfrac{1}{2}}\right) \geq -\sqrt{x^2 + \tfrac{1}{2}}\, g(x) + \sqrt{x^2 + \tfrac{1}{2}} - x \quad \text{if } 0 \leq x \leq \frac{\sqrt{2}}{2}.$$

Multiplying this last inequality by the nonnegative number $x/\sqrt{x^2 + 1/2}$ we obtain

$$(11) \quad \frac{x}{\sqrt{x^2 + \frac{1}{2}}} f\left(\sqrt{x^2 + \tfrac{1}{2}}\right) \geq -xg(x) + x - \frac{x^2}{\sqrt{x^2 + \frac{1}{2}}} \quad \text{if } 0 \leq x \leq \frac{\sqrt{2}}{2}.$$

Adding the inequalities (6) and (11), which now hold over the same range, yields

$$f(x) + \frac{x}{\sqrt{x^2 + \frac{1}{2}}} f\left(\sqrt{x^2 + \tfrac{1}{2}}\right) \geq x - \frac{x^2}{\sqrt{x^2 + \frac{1}{2}}} \quad \text{if } 0 \leq x \leq \frac{\sqrt{2}}{2},$$

whence

$$\int_0^{\frac{\sqrt{2}}{2}} f(x)dx + \int_0^{\frac{\sqrt{2}}{2}} \frac{x}{\sqrt{x^2 + \frac{1}{2}}} \; f\left(\sqrt{x^2 + \tfrac{1}{2}}\right) dx$$

$$\geq \int_0^{\frac{\sqrt{2}}{2}} \left(x - \frac{x^2}{\sqrt{x^2 + \frac{1}{2}}} \right) dx \; .$$

Upon making the appropriate change of variable in the second integral in the left-hand member above, and the same change in the right-hand member, we arrive at the desired inequality,

$$\int_0^1 f(x)dx \geq J \; .$$

In accordance with an earlier assertion about the multiplicity of solutions to the problem, we contend that the analysis indicates that if (f^*, g^*) is any pair of functions satisfying the following conditions, then (f^*, g^*) is a solution:

$$g_0(0) \leq g^*(0) \leq 1 + g_0\left(\frac{\sqrt{2}}{2}\right),$$

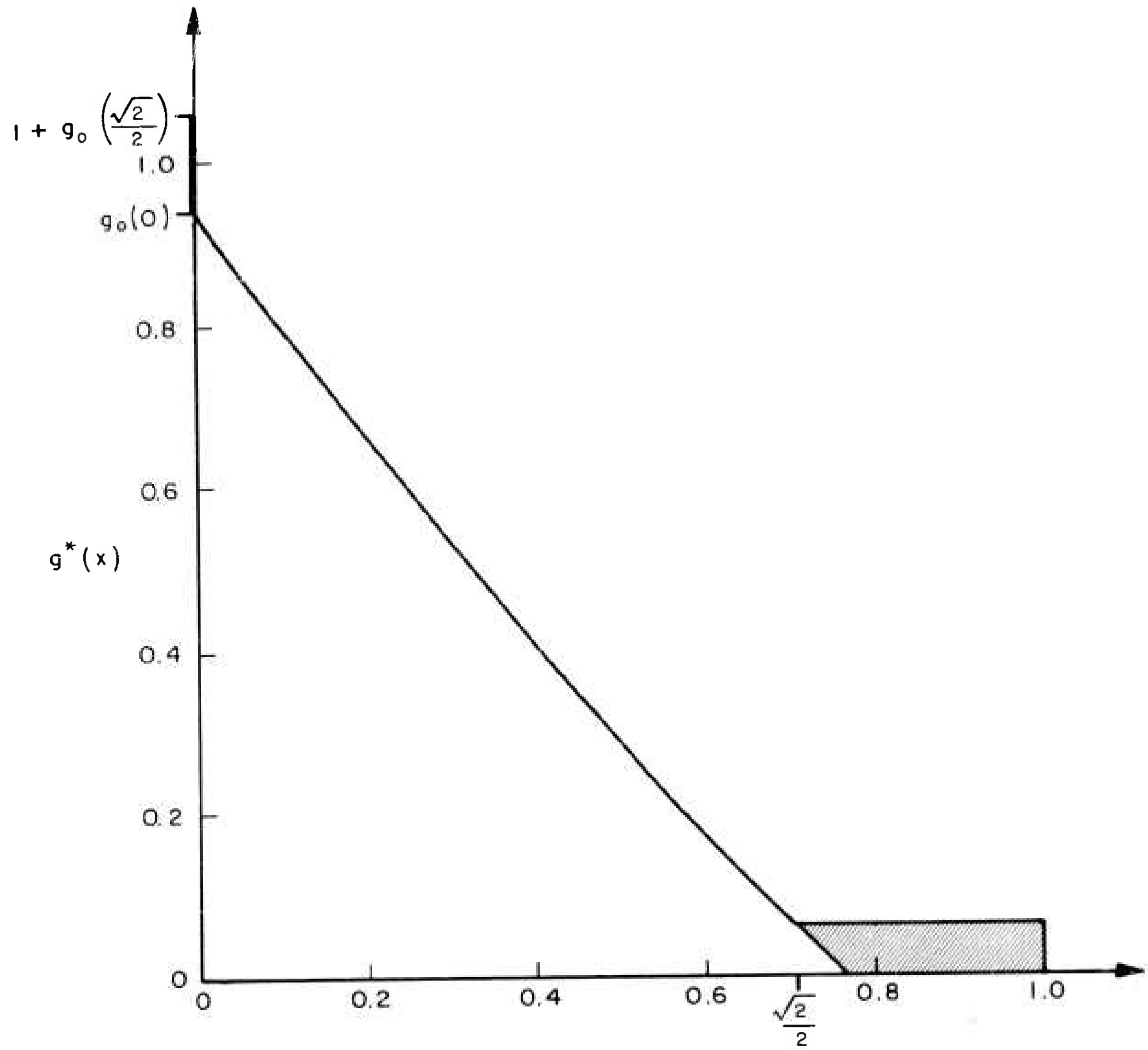
$$g^*(x) = g_0(x) \quad \text{if } 0 < x \leq \frac{\sqrt{2}}{2} \; ,$$

$$\max\left(0, \; g_0\left(\frac{\sqrt{2}}{2}\right) - x + \frac{\sqrt{2}}{2}\right) \leq g^*(x) \leq g_0\left(\frac{\sqrt{2}}{2}\right) \text{ if } \frac{\sqrt{2}}{2} < x \leq 1,$$

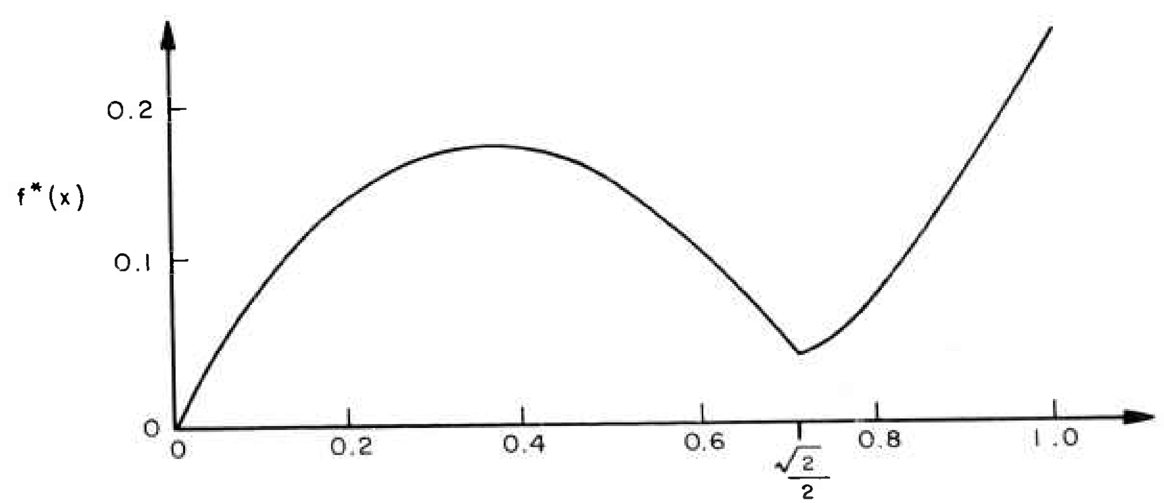
$$f^*(x) \geq f_0(x) \quad \text{with equality holding for almost all } x \in [0, 1] \; .$$

(It is conjectured that these are the only solutions to the problem. It has been the writer's experience in such matters, however, that a proof would perhaps be rather long and will not, therefore, be attempted in this paper.)

Graphs of these solutions showing the allowable variation of g^* are appended. Notice that there is a discontinuity in the derivative of f^* at $x = \frac{\sqrt{2}}{2}$.



$1 + g_0\left(\frac{\sqrt{2}}{2}\right)$
1.0
$g_0(0)$
0.8
0.6
$g^*(x)$
0.4
0.2
0
0
0.2
0.4
0.6
$\frac{\sqrt{2}}{2}$
0.8
1.0
x



0.2
$f^*(x)$
0.1
0
0
0.2
0.4
0.6
$\frac{\sqrt{2}}{2}$
0.8
1.0
x

LIST OF RAND NOTES ON LINEAR PROGRAMMING AND EXTENSIONS

RM-1264 Part 1: The Generalized Simplex Method for Minimizing a Linear Form under Linear Inequality Restraints, by G. B. Dantzig, A. Orden, and P. Wolfe, April 5, 1954. Published in the Pacific Journal of Mathematics, Vol. 5, No. 2, June, 1955, pp. 183-195. (ASTIA No. AD 114134)

RM-1265 Part 2: Duality Theorems, by G. B. Dantzig and A. Orden, October 30, 1953. (ASTIA No. AD 114135)

RM-1266 Part 3: Computational Algorithm of the Simplex Method by G. B. Dantzig, October 26, 1953. (ASTIA No. AD 114136)

RM-1267-1 Part 4: Constructive Proof of the Min-Max Theorem, by G. B. Dantzig, September 8, 1954. Published in the Pacific Journal of Mathematics, Vol. 6, No. 1, Spring, 1956, pp. 25-33. (ASTIA No. AD 114137)

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